Steady wave patterns on a non-uniform steady fluid flow

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A steady slightly non-uniform flow with a free surface is subject to a concentrated surface pressure which gives rise to a pattern of surface waves. (For gravity waves on deep water this is the well-known Kelvin ship-wave pattern.) The motion is assumed inviscid, and the waves are assumed small. A theory is developed for the wave pattern, based on the following assumptions:

(1) The stream velocity component normal to a wave crest is equal to the phase velocity based on the local wavelength;

(2) the separation between consecutive crests is equal to the local wavelength.

These assumptions are expressed in mathematical form, and the existence of a set of characteristic curves (associated with the group velocity) is deduced from them. These characteristics are not identical with the crests. Let the additional assumption be made that

(3) the characteristics all pass through the point disturbance; the characteristics are then completely defined and may be constructed by a step-by-step process starting at the point disturbance. The same construction gives the direction of the wave crests at all points. The wave crests can then be deduced.

Assumptions of the same type as (1) and (2) have long been familiar in various applications of ray tracing. For uniform flows the present theory gives the same pattern as the method of stationary phase.

1. Introduction

A steady flow with a free surface (the *basic flow*) is subject to a concentrated surface pressure which gives rise to a pattern of surface waves. The present paper is concerned with the problem of calculating this pattern. Methods of solution have long been known for the special case when the basic flow is uniform. The pattern can then be found approximately from fairly simple rules (see § 2 below). Examples may be found in Lamb's *Hydrodynamics* (Lamb 1932), where application is made to gravity waves (Kelvin's ship wave pattern) in § 256, and to capillary and combined capillary-gravity waves in § 272. By a change of coordinate system the problem of the uniform basic flow of constant depth becomes the problem of a pressure point travelling with constant velocity over fluid at rest. Pressure points travelling along curved paths on constant depth have been investigated by Stoker (1957, chap. 8).

Little seems to be known about the steady wave pattern when the basic flow is non-uniform. A problem of this type arose recently in the work of Sir Geoffrey

Taylor on thin films of water controlled by surface tension (Taylor 1959; equations and figures quoted from this paper will be preceded by the numeral II). Taylor studied wave patterns on uniform and axially symmetric flows, and an approximate theory which he developed for the latter agreed quite well with his measurements (see his figures II 16, 17.) It is the purpose of the present paper to give a more systematic theoretical discussion, based on different ideas, of such wave patterns on a slightly non-uniform stream.

In § 2 of the present paper the basic flow will be assumed uniform. Rules will be stated for the construction of approximate wave patterns from a point disturbance. For this problem the exact linearized solution may be found in the form of an oscillatory integral, valid in the whole field of flow. The rules follow on applying the principle of stationary phase to the exact solution, and are thus seen to give the wave pattern at a distance from the point disturbance.

In § 3 a new ray theory for slightly non-uniform flows will be derived. The equations for the wave crests will be deduced, not rigorously from the equations of motion (as for uniform flows), but from assumptions which appear physically reasonable and which resemble assumptions used in the familiar ray tracing of periodic water waves approaching a non-uniform shelving beach (Arthur, Munk & Isaacs 1952) and in other geophysical applications. The equations are used to construct a system of subsidiary curves, the *characteristic rays*, by a step-by-step method beginning at the disturbance. The crests can then be found by another step-by-step process. The present problem appears, however, more difficult to grasp physically than the problem of the shelving beach. In the latter there is an obvious invariant, the period of the wave, which gives an immediate physical meaning to the step-by-step calculation of the ray pattern on the beach. In our problem, on the other hand, all wavelengths are present, and the visible wavelength (except near the disturbance) depends on a stationary-phase condition. However, our formulation makes no explicit reference to stationary phase or to the allied notion of group velocity.

In § 4 the method will be applied to 'symmetric' waves on a thin capillary film which is either uniform or radially expanding, and equations for the crests will be derived. These approximate to Taylor's equations in the flow regions investigated by him, and are therefore confirmed by his experiments.

The distribution of amplitude in the wave pattern is not treated in the present paper. It will be assumed throughout that the motion is frictionless and irrotational, and that the wave amplitude is small.

2. Waves on a uniform stream

In this section it will be supposed that the basic flow is a uniform steady stream of velocity U, and that the small pressure giving rise to the waves is concentrated at a point. From this special case the waves due to a small pressure distributed over a finite area may be deduced by integration. Let rectangular Cartesian co-ordinates be taken as follows: The z-axis normal to the mean free surface, the x-axis parallel to the stream velocity, and the y-axis normal to the other two axes. The origin is taken so that $x = y = 0$ at the point disturbance. Polar co-ordinates are defined by $x = r\cos\theta$, $y = r\sin\theta$.

Our problem is to find the equations of the wave crests (and of all other curves of constant phase); more precisely, to find the equations of their projection on any plane z = const. In these equations the co-ordinate z therefore does not appear. (The variation of wave amplitude will not be treated in the present paper.) As has already been stated in the introduction, the solution is well known. The argument which is quite rigorous proceeds along the following lines: Let the pressure disturbance be of small magnitude so that the equations of motion can be linearized. Then the boundary-value problem for the disturbance can be solved explicitly everywhere. In particular, it is found that the surface deformation is of the form

$$\int dk \int d\gamma \, \frac{F(k,\gamma)\exp\{ik(x\sin\gamma - y\cos\gamma)\}}{c_p(k) - U\sin\gamma}$$

(with $F(k,\gamma)$ depending on the pressure disturbance). Here the resolution into plane waves is evident. Care is needed in interpreting this integral, since the denominator vanishes when $c_p(k) = U\sin\gamma$; this shows that the solution of the steady boundary-value problem is not unique. It is necessary now to choose the physically appropriate unique solution. For this purpose Lamb (1916) uses the 'Rayleigh viscosity' (see Lamb, 1932, § 242) which has no clear physical meaning but is mathematically convenient. It is physically more satisfactory to proceed differently and to consider the corresponding unsteady problem:

Suppose that the pressure disturbance appears at time $t = 0$, not at time $t = -\infty$. A unique unsteady wave motion results which can again be found explicitly in integral form and which tends after a long time to a limiting motion of the previous form but now with a well-defined meaning of the singular integral. This method usually leads to the same result as the Rayleigh viscosity. Examples of its use are given by Peters & Stoker (1957, pp. 174–81). The double integral, when the appropriate interpretation has been found, gives the wave pattern everywhere, subject merely to the conditions of linearization in an inviscid fluid. No additional approximation has been introduced. To obtain an approximate picture of the wave pattern without numerical computation, it is now assumed that x and y are large compared with some length scale of the problem. The double integral may then be approximated by the principle of stationary phase. It is shown by Lamb (1916) that the dominant contribution arises when

$$c_p(k) = U\sin\gamma,$$

and is in the form of a single integral

$$\int G(\gamma)\exp\{ik(\gamma)\,(x\sin\gamma - y\cos\gamma)\}\,d\gamma,$$

where $k(\gamma)$ is the solution of $c_p(k) = U\sin\gamma$ and where the range of integration depends on x and y. Clearly the points of stationary phase are the solutions of

$$\frac{d}{d\gamma}\{k(\gamma)\,(x\sin\gamma - y\cos\gamma)\} = 0.$$

If this equation has only one solution $\gamma_0(y/x)$ the crests are seen to be of the form

$$k\left(\gamma_0\left(\frac{y}{x}\right)\right)\left\{x\sin\gamma_0\left(\frac{y}{x}\right)-y\cos\gamma_0\left(\frac{y}{x}\right)\right\}=A$$

asymptotically when the distance from the origin is large, and are all similar with respect to the origin. This result can be expressed in a more convenient form for application by the following rules, valid at a large distance from the origin:

(1) The phase velocity $c_p(k)$, relative to the stream, of a regular two-dimensional wave train of wavelength $2\pi/k$ is found subject to the appropriate boundary conditions. Thus, $c_p(k) = \surd(g/k)$ for gravity waves on deep water (Lamb 1932, § 229, equation 6).

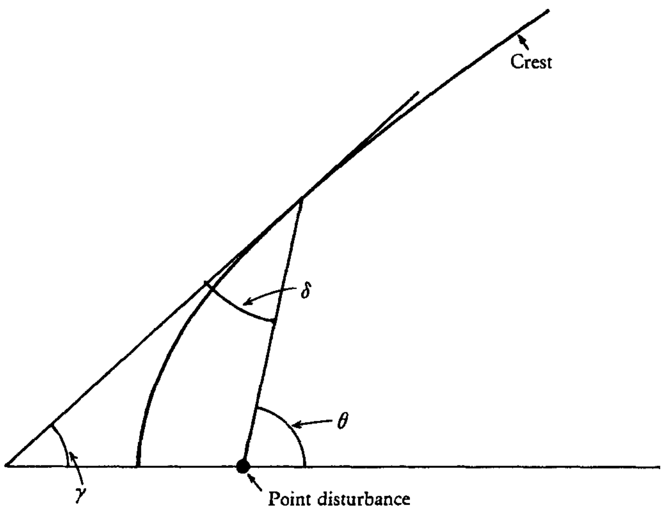

FIGURE 1. The angles γ, δ and θ.

(2) The angle $\gamma(k)$ is defined by

$$\sin\gamma = \frac{c_p(k)}{U}, \tag{2.1}$$

and the angle $\delta(\gamma, k) = \delta(k)$ by

$$\tan\delta = k\frac{d\gamma}{dk} = \frac{k}{c_p}\frac{dc_p}{dk}\tan\gamma. \tag{2.2}$$

(3) The radial function $r(k)$ and the angular function $\theta(k)$ are defined by

$$r(k) = \frac{A}{k\sin\delta(k)}, \quad \theta(k) = \gamma(k)+\delta(k), \tag{2.3}$$

where A is a constant.

Then, as k varies, these are parametric equations of a curve of constant phase. Different curves of constant phase correspond to different values of A. From one crest to the next, A changes by 2π. Evidently all curves of constant phase are similar with respect to the origin. It can be shown from the equations that $r\,d\theta/dr = -\tan\delta$ on a curve of constant phase, whence it follows that δ is the angle between the radius and the tangent, and γ is the angle between the tangent and the x-axis; see figure 1.

[The same equations as above are also obtained in the solution of the following problem:

To find the envelope of the family of straight lines

$$k(\gamma)\,(x\sin\gamma - y\cos\gamma) = A,$$

where $k(\gamma)$ is defined by the equation

$$c_p(k) = U\sin\gamma.]$$

The asymptotic treatment of the integral also gives an approximate expression for the amplitude; see Lamb (1932, § 256), Havelock (1908, § 14). When there are several points of stationary phase their contributions must be added. The derivation fails when the method of stationary phase fails in its simplest form, for example, near a line of cusps as in Kelvin's ship-wave pattern (Lamb 1932, § 256). The crests at a distance from the origin may then be found from the integral by more elaborate asymptotic methods. For Kelvin ship waves it has been shown in this way (Ursell 1960) that near the cusps a different law of similarity prevails. Here the kinematic method of the next section fails. For dispersive systems the method of stationary phase is the more fundamental one.

3. Waves on a slightly non-uniform stream

In the last section the stream velocity of the basic flow was assumed uniform. In the present section it will be assumed merely that the basic flow is irrotational, that it does not vary rapidly with distance, and that one velocity component (normal to the plane $z = 0$, say) is negligible throughout the fluid. (The free surface is then nearly parallel to $z = 0$.) An example of such a flow is the thin current sheet considered by Sir Geoffrey Taylor (1959, p. 309). To fix ideas, let it be assumed temporarily that the z-component of velocity vanishes on the plane $z = 0$. Let the x-axis and the y-axis be taken normal to each other and to the z-axis but otherwise arbitrary at this stage. Let the velocity components of the basic flow parallel to these axes be denoted by $U(x,y)$ and $V(x,y)$, respectively; these vary only slowly with x and y, and in addition it is assumed that their dependence on z is negligible. Let the equation of the free surface of the basic flow be $z = h(x,y)$.

Suppose, as in the last section, that the flow is slightly disturbed by a small concentrated pressure which gives rise to the wave pattern. It will be supposed that through every point (x,y) of the free surface there passes just one phase curve, for example, a crest or trough; this restriction can easily be removed. It will also be supposed that (except near the disturbance) the phase curves are not strongly curved. Then a local wavelength $2\pi/k(x,y)$ can be defined approximately as the distance between neighbouring wave crests. Let $\gamma(x,y)$ denote the angle between the x-axis and the tangent at (x,y) to the phase curve through (x,y), and let $\partial/\partial s$ and $\partial/\partial n$ denote differentiation along and normal to a phase curve respectively,

$$\frac{\partial}{\partial n} = \cos\gamma\frac{\partial}{\partial y} - \sin\gamma\frac{\partial}{\partial x},$$

$$\frac{\partial}{\partial s} = \cos\gamma\frac{\partial}{\partial x} + \sin\gamma\frac{\partial}{\partial y}.$$

Consider first the phase velocity $c_p(k, h)$ relative to a *uniform* stream of *constant* depth h, of a regular two-dimensional wave train of wavelength $2\pi/k$ subject to the appropriate boundary conditions (compare § 2 above). Then our construction of the phase curves is based on the following two equations

$$c_p\{k(x,y), h(x,y)\} = U(x,y)\sin\gamma(x,y) - V(x,y)\cos\gamma(x,y), \tag{3.1}$$

$$\frac{\partial}{\partial n}\gamma(x,y) = -\frac{1}{k(x,y)}\frac{\partial}{\partial s}k(x,y). \tag{3.2}$$

Equation (3.1) corresponds to equation (2.1) above. It states that in a steady wave pattern the stream velocity normal to the crest is balanced by the phase velocity, and also that the phase velocity relative to a slightly non-uniform stream and on variable depth is adequately approximated by the phase velocity obtained from constant-depth theory. Equation (3.2) is already familiar in the geometry of ray systems (see, e.g. Longuet-Higgins 1957, equation 10). It states that the distance between neighbouring phase curves is proportional to the wavelength. To prove this, consider any family of plane curves (here the phase curves), expressed in the form $f(x,y) = a$, where a is a variable parameter. Then it can be shown by elementary differential geometry that

$$\frac{\partial\gamma}{\partial n} = \frac{\partial}{\partial s}\ln\left(\frac{1}{|\operatorname{grad} f|}\right);$$

the proof is omitted. The perpendicular distance between the neighbouring curves $f(x,y) = a$ and $f(x,y) = a + da$ is easily shown to be $da/|\operatorname{grad} f|$, and if this is taken to be proportional to the wavelength, equation (3.2) follows.

As the first stage in the construction of the phase curves, let us find a differential equation for the slope angle $\gamma(x,y)$. To this end, let (3.1) be differentiated with respect to x and to y

$$\frac{\partial c_p}{\partial k}\frac{\partial k}{\partial x} + \frac{\partial c_p}{\partial h}\frac{\partial h}{\partial x} = \frac{\partial U}{\partial x}\sin\gamma - \frac{\partial V}{\partial x}\cos\gamma + (U\cos\gamma + V\sin\gamma)\frac{\partial\gamma}{\partial x}, \tag{3.3}$$

$$\frac{\partial c_p}{\partial k}\frac{\partial k}{\partial y} + \frac{\partial c_p}{\partial h}\frac{\partial h}{\partial y} = \frac{\partial U}{\partial y}\sin\gamma - \frac{\partial V}{\partial y}\cos\gamma + (U\cos\gamma + V\sin\gamma)\frac{\partial\gamma}{\partial y}; \tag{3.4}$$

and let (3.2) be written in Cartesian co-ordinates

$$\cos\gamma\frac{\partial\gamma}{\partial y} - \sin\gamma\frac{\partial\gamma}{\partial x} = -\frac{1}{k}\left(\cos\gamma\frac{\partial k}{\partial x} + \sin\gamma\frac{\partial k}{\partial y}\right). \tag{3.5}$$

Thus

$$\frac{\partial}{\partial x}(k\cos\gamma) + \frac{\partial}{\partial y}(k\sin\gamma) = 0$$

i.e. the *vector wave-number* $\mathbf{k} = (-k\sin\gamma, k\cos\gamma)$ is irrotational. The line integral $\int\mathbf{k}.d\mathbf{s}$ taken between two points of the flow field is therefore independent of the path of integration and may be interpreted as the phase difference between the two points. A very interesting alternative development of wave patterns, starting from the phase function, is given by Whitham (1960) in the following paper. His treatment also covers the extension to three dimensions and to certain unsteady wave patterns.

After this digression, if now $\partial k/\partial x$ and $\partial k/\partial y$ are eliminated from equations (3.3), (3.4) and (3.5), and if the function $c_g(k,h)$ is defined by

$$c_g(k,h) = c_p(k,h) + k\frac{\partial}{\partial k}c_p(k,h) \tag{3.6}$$

(in agreement with the usual definition of group velocity), then it is found that

$$\begin{aligned}\{U(x,y) - c_g(k,h)\sin\gamma\}\frac{\partial\gamma}{\partial x} + \{V(x,y) + c_g(k,h)\cos\gamma\}\frac{\partial\gamma}{\partial y} \\ = -\cos\gamma\left(\frac{\partial U}{\partial x}\sin\gamma - \frac{\partial V}{\partial x}\cos\gamma\right) - \sin\gamma\left(\frac{\partial U}{\partial y}\sin\gamma - \frac{\partial V}{\partial y}\cos\gamma\right) \\ + \frac{\partial c_p}{\partial h}\left(\frac{\partial h}{\partial x}\cos\gamma + \frac{\partial h}{\partial y}\sin\gamma\right).\end{aligned} \tag{3.7}$$

Here $U(x,y)$, $V(x,y)$, $h(x,y)$ and their derivatives are known functions of x and y, while $\partial c_p/\partial h$ and c_g are known functions of k and h and therefore, by (3.1), of x, y and γ. Thus equation (3.6) is a quasi-linear equation of the first order for γ, of the form

$$F_1(x,y,\gamma)\frac{\partial\gamma}{\partial x} + F_2(x,y,\gamma)\frac{\partial\gamma}{\partial y} = F_3(x,y,\gamma), \tag{3.8}$$

where F_1, F_2 and F_3 are known functions. Solutions of this equation have the following property.

If γ is given at a single point: $\gamma(x_0, y_0) = \gamma_0$, say; then equation (3.7) defines γ uniquely along a curve through (x_0, y_0), the *characteristic curve*, which depends on γ_0.

This result is proved by Courant & Hilbert (1937, pp. 51–4) and appears reasonable from the form of (3.8). For at (x_0, y_0) the values of F_1, F_2 and F_3 are known; thus (3.8) states that the rate of change of γ in a certain known direction is known. On proceeding an infinitesimal distance in this direction one finds new values of x, y and γ, and the result follows when this process is repeated indefinitely. It is noteworthy that the characteristic direction is along the resultant of the group velocity, taken normal to the phase curve, and of the velocity of the basic flow.

So far the theory has been developed in terms of the inclination $\gamma(x,y)$. This is not necessarily the most convenient dependent variable. Similar equations can be derived for the variation of $k(x,y)$ along a characteristic, and thence for the variation of any function of x, y, γ and k. In particular, the following equations can be obtained

$$\begin{aligned}\{U(x,y) - c_g(k,h)\sin\gamma\}\frac{\partial}{\partial x}(k\cos\gamma) + \{V(x,y) + c_g(k,h)\cos\gamma\}\frac{\partial}{\partial y}(k\cos\gamma) \\ = k\left(\frac{\partial U}{\partial y}\sin\gamma - \frac{\partial V}{\partial y}\cos\gamma\right) - k\frac{\partial c_p}{\partial h}\frac{\partial h}{\partial y},\end{aligned} \tag{3.9}$$

$$\begin{aligned}\{U(x,y) - c_g(k,h)\sin\gamma\}\frac{\partial}{\partial x}(k\sin\gamma) + \{V(x,y) + c_g(k,h)\cos\gamma\}\frac{\partial}{\partial y}(k\sin\gamma) \\ = -k\left(\frac{\partial U}{\partial x}\sin\gamma - \frac{\partial V}{\partial x}\cos\gamma\right) + k\frac{\partial c_p}{\partial h}\frac{\partial h}{\partial x},\end{aligned} \tag{3.10}$$

$$\{U(x,y)-c_g(k,h)\sin\gamma\}\frac{\partial}{\partial x}(kx\cos\gamma+ky\sin\gamma)$$
$$+\{V(x,y)+c_g(k,h)\cos\gamma\}\frac{\partial}{\partial y}(kx\cos\gamma+ky\sin\gamma)$$
$$=k\sin\gamma\left(x\frac{\partial U}{\partial y}-y\frac{\partial U}{\partial x}\right)-k\cos\gamma\left(x\frac{\partial V}{\partial y}-y\frac{\partial V}{\partial x}\right)$$
$$-k\frac{\partial c_p}{\partial h}\left(x\frac{\partial h}{\partial y}-y\frac{\partial h}{\partial x}\right)+k(U\cos\gamma+V\sin\gamma). \qquad (3.11)$$

In these equations the point $x = y = 0$ is an arbitrary origin of co-ordinates, not necessarily coincident with the point disturbance. Let us consider the most important special case when U = constant; $V = 0$; and h = constant, previously treated in § 2. Then the right-hand side of (3.7) vanishes, and so γ is constant along a characteristic. Similarly k is constant along a characteristic, and it follows that F_1 and F_2 are also constant along a characteristic. Thus for a uniform basic flow the characteristics are all straight lines, but their position in the field of flow is still unknown.

We have seen that a complete characteristic curve can be constructed if γ is known at one point on it, but we have not yet seen how such initial values of γ are to be assigned, nor has the location of the point disturbance entered into the calculation. For these reasons the characteristic rays are not yet completely determined. We now introduce the following:

Assumption *A*. *The characteristic curves of equation* (3.8) *all pass through the disturbance.*

This assumption is certainly valid for uniform basic flows. This follows at once from the theory of § 2 where it was shown that γ is constant along straight lines through the disturbance. Alternatively, we may consider points on the streamline through the disturbance; on these, by symmetry, γ is either 0 or $\frac{1}{2}\pi$. If characteristics pass through such points, γ is 0 or $\frac{1}{2}\pi$ in a region (from equations (3.9) and (3.10)). If this conclusion is rejected as absurd, it again follows that for uniform basic flows the characteristics all pass through disturbance. (Near the disturbance the validity of (3.1) and (3.2) is in any case dubious.) Assumption *A* is extended to non-uniform basic flows by analogy.

It is now easily seen how the characteristic curves can be constructed. To each arbitrary initial value of γ at the disturbance there corresponds a characteristic of (3.7) through the disturbance. Thus γ is defined uniquely on each characteristic, and to each point of that part of the (x, y) plane covered by characteristics there corresponds a value of γ (or possibly several values of γ).

It is now seen how the phase curves (such as the crests) can be constructed. For $\gamma(x, y)$ which has just been obtained is the angle between the x-axis and the tangent to a phase curve. Thus the phase curves are the solutions of

$$\frac{dy}{dx}=\tan\gamma(x,y)$$

which may be integrated by known methods. Thus the construction of the phase curves is now completed in principle.

In this way one can obtain by a different method the rules for a uniform basic flow which were stated and explained in § 2. As in § 2, suppose that the origin of co-ordinates is taken at the point disturbance, that h = const., that U = const., and $V = 0$, and that $x = r\cos\theta$ and $y = r\sin\theta$. Then we have seen that the characteristics are straight lines through the origin, on which γ is constant. Thus, from the characteristic θ = const.

$$\tan\theta = \frac{c_g(k)\cos\gamma}{U - c_g(k)\sin\gamma},$$

where

$$c_p(k) = U\sin\gamma$$

and

$$c_g(k) = c_p(k) + k\frac{\partial c_p}{\partial k} = U\sin\gamma + k\frac{\partial c_p}{\partial k}.$$

Thus, by straightforward substitution

$$\tan(\theta-\gamma) = \frac{\tan\theta - \tan\gamma}{1+\tan\theta\tan\gamma} = \frac{c_g - U\sin\gamma}{U\cos\gamma} = \frac{k\frac{\partial c_p}{\partial k}}{U\cos\gamma} = k\frac{\partial\gamma}{\partial k}$$
$$= \tan\delta(k),$$

where $\delta(k)$ is defined by (2.2). Thus $\theta(k) = \gamma(k) + \delta(k)$, as in (2.3), and so $\delta(k)$, as appears from figure 1, is the angle between the radius and the tangent. It follows that

$$r\frac{d\theta}{dr} = -\tan\delta(k),$$

whence

$$\frac{1}{r}\frac{dr}{dk} = -\frac{1}{\tan\delta}\frac{d\theta}{dk} = -\frac{1}{\tan\delta}\left(\frac{d\gamma}{dk} + \frac{d\delta}{dk}\right)$$
$$= -\frac{1}{k} - \frac{1}{\tan\delta}\frac{d\delta}{dk}.$$

By integration,

$$\ln r = -\ln k - \ln\sin\delta(k) + \text{const.},$$

which agrees with the radial equation (2.3). Thus the procedure of the present section reduces to the rules of § 2 for the special case of a uniform basic flow.

Next let us briefly consider flows with radial symmetry, such as the expanding sheet studied by Sir Geoffrey Taylor (1959, p. 309). Let another system of polar co-ordinates (R, Θ) relative to the centre of symmetry be introduced. Then the velocity components of this type of basic flow are of the form $U = C(R)\cos\Theta$, $V = C(R)\sin\Theta$, and the depth is of the form $h = h(R)$. If by definition

$$X = R\cos\Theta, \quad Y = R\sin\Theta,$$

then

$$X\frac{\partial}{\partial Y} - Y\frac{\partial}{\partial X} = \frac{\partial}{\partial\Theta}.$$

It is easy to see that the right-hand side of (3.11) vanishes since

$$\frac{\partial U}{\partial\Theta} = -V, \quad \frac{\partial V}{\partial\Theta} = U,$$

and therefore

$$kX\cos\gamma + kY\sin\gamma = kR\cos(\Theta-\gamma) \tag{3.12}$$

is constant along characteristics in flows with radial symmetry. The physical meaning of this invariant is not obvious.

4. Applications

We consider symmetrical capillary waves in a thin two-sided sheet. Sir Geoffrey Taylor (1959, equation II 4) has shown that in a sheet of uniform thickness $2h$

$$c_p(k) = \{(T/\rho)\, k \tanh kh\}^{\frac{1}{2}}$$
$$\sim (Th/\rho)^{\frac{1}{2}}\, k \quad \text{as} \quad kh \to 0.$$

Now kh is small when θ is small (behind the disturbance) see (4.3) below. In this region the error will therefore be small, while the equations for the wave pattern will simplify considerably if the approximate relation

$$c_p(k) = (Th/\rho)^{\frac{1}{2}}\, k \tag{4.1}$$

is taken to be exact, as will be done in the calculations that follow. (Compare Taylor 1959, equation II 24.)

Let us suppose first that the basic flow is uniform and that U is constant. From (2.1)

$$\sin\gamma = (T/\rho U^2 h)^{\frac{1}{2}}\, kh = W^{\frac{1}{2}} kh,$$

where $W < 1$ is the Weber number of the flow. From (2.2),

$$\tan\delta = \tan\gamma, \quad \text{whence} \quad \delta = \gamma, \quad \text{and} \quad \sin\delta = \sin\gamma = W^{\frac{1}{2}} kh.$$

From (2.3)

$$r(k) = \frac{A}{k \sin\delta(k)} = \frac{A}{W^{\frac{1}{2}} k^2 h} \tag{4.2}$$

and

$$\theta(k) = \gamma(k) + \delta(k) = 2\gamma(k).$$

Thus

$$\sin\tfrac{1}{2}\theta = \sin\gamma = W^{\frac{1}{2}} kh. \tag{4.3}$$

By elimination of k from (4.2) and (4.3) it is found that

$$W^{-\frac{1}{2}}(r/h)\sin^2\tfrac{1}{2}\theta = A, \tag{4.4}$$

a set of parabolas; if only the leading term for small θ is retained, this is

$$W^{-\frac{1}{2}}(r\theta^2/4h) = A. \tag{4.5}$$

A photograph of the wave pattern is shown by Taylor, Figure II 14. Equation (4.4) is identical with equation II 32. In fact the approximation cannot consistently be carried beyond (4.5) if the phase velocity is taken in the approximate form (4.1). The exact velocity relation may be used with little trouble to give higher corrections when θ is small, but this will not be done here.

Let us suppose next that there is radial symmetry; that the radial velocity is constant, $C(R) = C_0$, say (cf. the end of § 3); and that the depth h varies as R^{-1}. These conditions are satisfied by an expanding sheet, (see II, p. 299). If it is still assumed that

$$c_p(k) = (Th/\rho)^{\frac{1}{2}}\, k$$

exactly, then the wave pattern can be found in explicit form, as will now be shown.

The characteristic rays are first found. Let ψ denote the angle between the characteristic through (R, Θ) and the radius $\Theta = 0$. Then, from the equation for the characteristic,

$$\tan\psi = \frac{C_0 \sin\Theta + c_g(k,h)\cos\gamma}{C_0\cos\Theta - c_g(k,h)\sin\gamma}, \tag{4.6}$$

whence
$$\tan(\psi-\Theta) = \frac{c_g\cos(\gamma-\Theta)}{C_0-c_g\sin(\gamma-\Theta)}. \tag{4.7}$$

Also
$$c_g = c_p+k(\partial c_p/\partial k) = 2c_p, \quad \text{from (4.1);} \tag{4.8}$$

(it is the simplicity of this relation which simplifies the calculation) and
$$c_p(k) = C_0\sin(\gamma-\Theta), \quad \text{from (3.1).} \tag{4.9}$$

Thus
$$\tan(\psi-\Theta) = \frac{2\sin(\gamma-\Theta)\cos(\gamma-\Theta)}{1-2\sin^2(\gamma-\Theta)}$$
$$= \tan 2(\gamma-\Theta), \tag{4.10}$$

whence
$$\psi = 2\gamma-\Theta. \tag{4.11}$$

Moreover, from (3.12), $kR\cos(\gamma-\Theta)$ is constant along a characteristic, and from (4.9), since h varies as R^{-1}, $R^{\frac{1}{2}}k^{-1}\sin(\gamma-\Theta)$ is constant everywhere. On multiplying these, we find that $R^{\frac{3}{2}}\sin 2(\gamma-\Theta)$ is constant along a characteristic. If the point disturbance is situated at $(R_0, 0)$ and if γ_0 is the initial value γ on a characteristic,
$$R^{\frac{3}{2}}\sin 2(\gamma-\Theta) = R_0^{\frac{3}{2}}\sin 2\gamma_0. \tag{4.12}$$

By elementary differential geometry on a characteristic,
$$\begin{aligned} R\frac{d\Theta}{dR} &= \tan(\psi-\Theta) \\ &= \tan 2(\gamma-\Theta) \quad \text{from (4.10),} \\ &= \left(\frac{R_0}{R}\right)^{\frac{3}{2}}\sin 2\gamma_0\{1-(R_0/R)^3\sin^2 2\gamma_0\}^{-\frac{1}{2}} \quad \text{from (4.12).} \end{aligned} \tag{4.13}$$

Thus
$$\begin{aligned} \Theta &= \int_{R_0}^{R}\frac{dR}{R}\left(\frac{R_0}{R}\right)^{\frac{3}{2}}\sin 2\gamma_0\{1-(R_0/R)^3\sin^2 2\gamma_0\}^{-\frac{1}{2}} \\ &= -\frac{2}{3}\int_1^v\frac{dv}{v}\,v\sin 2\gamma_0(1-v^2\sin^2 2\gamma_0)^{-\frac{1}{2}}, \quad \text{where} \quad v = (R_0/R)^{\frac{3}{2}} \\ &= -\tfrac{2}{3}\sin^{-1}(v\sin 2\gamma_0)+\tfrac{4}{3}\gamma_0, \end{aligned}$$

and the equation of the characteristic rays is thus
$$\sin(2\gamma_0-\tfrac{3}{2}\Theta) = v\sin 2\gamma_0 = \left(\frac{R_0}{R}\right)^{\frac{3}{2}}\sin 2\gamma_0, \tag{4.14}$$

where γ_0 is the initial value of γ at the disturbance $(R_0, 0)$. From (4.13),
$$\tan 2(\gamma-\Theta) = \sin(2\gamma_0-\tfrac{3}{2}\Theta)\{1-\sin^2(2\gamma_0-\tfrac{3}{2}\Theta)\}^{-\frac{1}{2}} = \tan(2\gamma_0-\tfrac{3}{2}\Theta),$$

whence
$$\gamma-\Theta = \gamma_0-\tfrac{3}{4}\Theta. \tag{4.15}$$

It is now possible to find the slope of the phase curve through any point (R, Θ). First the parameter γ_0 of the ray through (R, Θ) is found from (4.14), then the slope γ is found from (4.15).

The differential equation of the crests is
$$\begin{aligned} R\frac{d\Theta}{dR} &= \tan(\gamma-\Theta) \\ &= \tan(\gamma_0-\tfrac{3}{4}\Theta) \end{aligned} \tag{4.16}$$

from (4.15), where $\gamma_0(R, \Theta)$ is the function defined implicitly by (4.14)

$$\tan 2\gamma_0 = \frac{\sin \frac{3}{2}\Theta}{\cos \frac{3}{2}\Theta - (R_0/R)^{\frac{3}{2}}} = \frac{\sin \frac{3}{2}\Theta}{\cos \frac{3}{2}\Theta - v}. \tag{4.17}$$

It is easy to show that

$$\tan (2\gamma_0 - \tfrac{3}{2}\Theta) = \frac{v \sin \frac{3}{2}\Theta}{1 - v\cos \frac{3}{2}\Theta}$$

$$= \frac{2R(d\Theta/dR)}{1 - R^2(d\Theta/dR)^2} \quad \text{from (4.16).}$$

This is a quadratic equation for $R(d\Theta/dR)$; the required root is

$$R\frac{d\Theta}{dR} = -\tfrac{3}{2}v\frac{d\Theta}{dv} = -\frac{1 - v\cos \frac{3}{2}\Theta}{v \sin \frac{3}{2}\Theta} + \left(\frac{1 - 2v\cos \frac{3}{2}\Theta + v^2}{v^2 \sin^2 \frac{3}{2}\Theta}\right)^{\frac{1}{2}}. \tag{4.18}$$

This root tends to zero as v tends to zero, compare (4.16) and (4.17). If now $\cos \frac{3}{2}\Theta$ is chosen as a new variable instead of Θ, it is possible to integrate (4.18). The solution is

$$\frac{1}{v} - \left(\frac{1}{v^2} - \frac{2\cos \frac{3}{2}\Theta}{v} + 1\right)^{\frac{1}{2}} = \text{const.},$$

as may be verified by differentiation.

The phase curves are therefore

$$\left(\frac{R}{R_0}\right)^{\frac{3}{2}} - \left\{\left(\frac{R}{R_0}\right)^3 - 2\left(\frac{R}{R_0}\right)^{\frac{3}{2}} \cos \tfrac{3}{2}\Theta + 1\right\}^{\frac{1}{2}} = \text{const.}, \tag{4.19}$$

where (R, Θ) are polar co-ordinates relative to the centre of the basic streaming flow, and the centre of disturbance is at $(R, 0)$. To determine which phase curves are crests it is necessary to find the actual phase, such that the distance between successive crests is equal to the local wavelength as given by (4.1). The phase might possibly be a complicated function of the left-hand side of (4.19). It is easily found by taking R large on the phase curves (which are almost straight at infinity) and comparing with (4.1). It appears that the phase ϵ is in fact a linear function

$$\epsilon = B\left[\left\{\left(\frac{R}{R_0}\right)^3 - 2\left(\frac{R}{R_0}\right)^{\frac{3}{2}} \cos \tfrac{3}{2}\Theta + 1\right\}^{\frac{1}{2}} + 1 - \left(\frac{R}{R_0}\right)^{\frac{3}{2}}\right], \tag{4.20}$$

where
$$B = \tfrac{1}{3}R_0^{\frac{3}{2}}\left(\frac{4\pi\rho C_0^3}{TQ}\right)^{\frac{1}{2}}, \quad Q = 4\pi R C_0 h$$

is the volume flux, and ϵ has been made to vanish when $\Theta = 0$ and $R > R_0$. It is clearly legitimate for the purpose of this argument to allow R to tend to infinity, although in fact an instability associated with a different type of wave causes the thin film to break at a finite radius. It is easy to see that

$$B = \frac{1}{3}\frac{R_0^{\frac{3}{2}}}{R_1^{\frac{1}{2}}}\frac{1}{h_1}, \tag{4.21}$$

where R_1 is the radius at which $W = (T/\rho C_0^2 h_1) = 1$, and $2h_1$ is the thickness there.

To obtain an approximation for the phase curves downstream when the phase ϵ is small, we have from (4.20) that

$$\left\{\left(\frac{1}{v}-1\right)^2+\frac{2}{v}\left(1-\cos\tfrac{3}{2}\Theta\right)\right\}^{\frac{1}{2}}-\left(\frac{1}{v}-1\right)=\frac{\epsilon}{B}$$

exactly where $v = (R_0/R)^{\frac{3}{2}} < 1$. The left-hand side for small θ is

$$\begin{aligned}&(1-\cos\tfrac{3}{2}\Theta)/(1-v)+O(v\Theta^4)\\ &\quad=\tfrac{9}{8}\Theta^2/(1-v)\end{aligned}$$

approximately. Thus

$$\begin{aligned}\Theta^2&=\frac{8\epsilon}{9B}(1-v)\\ &=\frac{8}{3}\frac{\epsilon h_1}{R_1}\left[\left(\frac{R_1}{R_0}\right)^{\frac{3}{2}}-\left(\frac{R_1}{R}\right)^{\frac{3}{2}}\right] \qquad (4.22)\end{aligned}$$

is the approximate polar equation of the curve of phase ϵ, where R is the distance from the centre of the basic flow, R_1 is the outer radius where $W = 1$, R_0 is the distance of the disturbance from the centre of the basic flow, and $2h_1$ is the thickness of the film at distance R_1. Equation (4.22) agrees with Taylor's result (II 42) which is illustrated in figures II 16 and 17 of his paper, though in the present calculation it is not assumed that the crests pass near the disturbance. Instead it has been assumed that the characteristic rays pass through the disturbance, and this is sufficient to fix the position of the phase curves. There is no difficulty in principle in carrying out a similar calculation for the exact phase-velocity relation, but a method of successive approximation is then needed near $\Theta = 0$.

5. Discussion and conclusions

It was assumed at the beginning of § 3 that the velocity component normal to $z = 0$ vanishes on $z = 0$, but it will be clear that this assumption was not fully used. All that is needed is the phase-velocity $c_p(k, h)$ as a function of wave-number k, with the local depth h entering as a parameter. (Taylor's 'anti-symmetric' waves, equation II 9, are an example of this.) And if it is some parameter other than depth which changes slowly from point to point then the theory is unchanged, provided that the phase velocity can still be found as a function of k and of the new parameter. Nor is it necessary that the disturbance should be caused by a point pressure rather than by some other concentrated physical agency.

The theory of § 3 is based on physically plausible assumptions, but no rigorous attempt has yet been made to find sufficient conditions for its validity. It would be helpful if explicit solutions could be found for some non-uniform basic flows, perhaps with axial symmetry. A comparison with the approximate theory would then become possible. It would also be helpful if the present theory could be made to appear as the first stage in a well-defined scheme of successive approximation, so that the errors could be estimated.

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